VOLUME 77 SEPARATE NO. 59

PROCEEDINGS

AMERICAN SOCIETY OF CIVIL ENGINEERS

February, 1951

LIMIT DESIGN OF BEAMS AND FRAMES

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STRUCTURAL DIVISION

Headquarters of the Society
33 W. 39th St.
New York 18, N.Y.

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Proceedings—Vol. 77 February, 1951 Separate No. 59

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

LIMIT DESIGN OF BEAMS AND FRAMES

BY H. J. GREENBERG[1] AND W. PRAGER,[2] M. ASCE

SYNOPSIS

The paper is concerned with the limit design of statically indeterminate beams or frames under the action of given loads. The limit moments under which individual sections act as yield hinges are supposed to be known throughout the structure and the safety factor against collapse is sought. This safety factor is defined as the multiple to which the actual loads must be increased before yield hinges develop in a number sufficient to render the structure unstable. Two extremum principles are established for the safety factor. With these principles two estimates of the safety factor can be obtained, one of which is known to be too small whereas the other is known to be too large. The gap between these estimates can be reduced by a repeated application of the extremum principles.

1. INTRODUCTION

As applied to statically indeterminate beams and frames, the method of limit design[3] is based on the assumption that the relation between the bending moment M and the curvature c of an elastic-plastic beam is of the type shown in Fig. 1. The absolute value of the curvature can increase indefinitely under the constant bending moments $\pm M_o$. The shape of the transition curve AB between the fully elastic branch OA and the fully plastic branch BC of the curve is immaterial for the application of limit design.

The fact that the absolute value of the limit moment (M_o in Fig. 1) is the same regardless of the sign of the moment means that it is necessary to consider only absolute values of the bending moments. As a rule, the value of the bending moment on a member will equal the limit moment only for discrete locations along the member. At these cross sections "yield hinges" are formed,

NOTE.—Written comments are invited for publication; the last discussion should be submitted by August 1, 1951.

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[3] "Theory of Limit Design," by J. A. Van den Broek, John Wiley & Sons, Inc., New York, N. Y., 1948.

which reduce the degree of indeterminacy of the structure. A sufficient number of yield hinges transforms the structure (or part of it) into a mechanism. To test whether this stage has been reached, it is sufficient to replace all yield hinges by perfect hinges and to consider the structural members which are joined by these hinges as completely rigid. If the structure modified in this manner is capable of deformation, however small, despite the assumed rigidity of its members, the actual appearance of the yield hinges under consideration would transform the structure into a mechanism and lead to the collapse of the structure. For instance, the frame shown in Fig. 2 becomes a mechanism if yield hinges develop at the seven cross sections marked A, B, $\cdots$, G in Fig. 3. If these yield hinges are replaced by perfect hinges and the members between these hinges are considered rigid, the system is capable of the type of deformation indicated by dotted lines in Fig. 3.

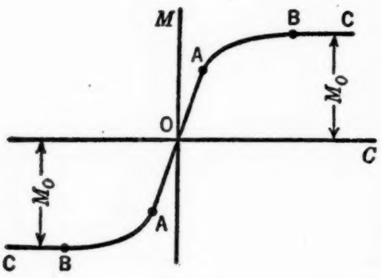

FIG. 1

The limit design problems considered in this paper are concerned with statically indeterminate beams or frames under the action of given loads. The limit moments are known throughout the structure, and the safety factor s against collapse is sought. This safety factor is defined as the multiple to which the actual loads must be increased before yield hinges develop in a

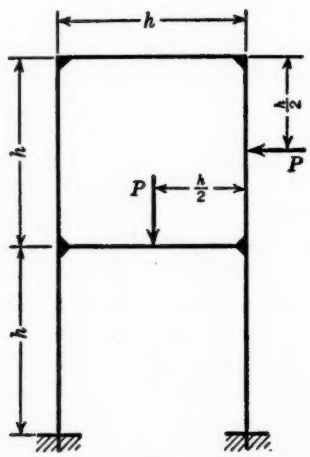

FIG. 2

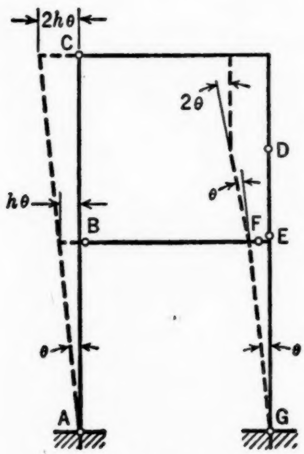

FIG. 3

number sufficient to transform the structure into a mechanism, thus precipitating collapse.

At first glance, a problem of this kind would seem to require a detailed investigation of the elastic-plastic behavior of the structure under gradually

increasing loads. This laborious investigation is not necessary, however, if only the safety factor against collapse is required. It is this fact which makes the method of limit design so attractive. Although in most cases the exact value of the safety factor cannot be computed directly, the following sections of this paper offer principles from which lower and upper limiting values ("bounds") for the safety factor can be obtained.

2. A Lower Bound for the Safety Factor

The principle to be established in this section furnishes the lower bound for the safety factor against collapse. Any such lower bound may be considered as an estimate of the safety factor, which can err only on the conservative side.

Consider a statically indeterminate beam or frame under the action of given loads. Any bending moment distribution that is computed from these loads, and from arbitrarily assigned values of the statically indeterminate quantities, will be said to be "statically compatible" with the given loads. Any bending moment distribution such that at no cross section does the bending moment exceed the limit moment for the cross section will be called an "admissible" distribution of bending moments.

The given loads (design loads) are not expected to cause collapse. Consider, now, the system of loads obtained by multiplying each of the given loads by the same multiplier. This multiplier will be called "statically admissible," if there exists an admissible distribution of bending moments which is statically compatible with the increased loads.

With this terminology the principle for the safety factor can be formulated as follows:

> Theorem I.—The safety factor against collapse is the largest statically admissible multiplier.

A proof of the principle stated in theorem I is given in the Appendix. As an illustration of the manner in which this principle is used, consider the indeterminate beam shown in Fig. 4(a). For simplicity, assume that the limit moment M_o is constant along the beam. If the statically indeterminate clamping couples X and Y are known, the bending moments of this beam are obtained by superimposing: (a) the bending moments which the given loads would produce if the beam were simply supported at its ends; and (b) the bending moments

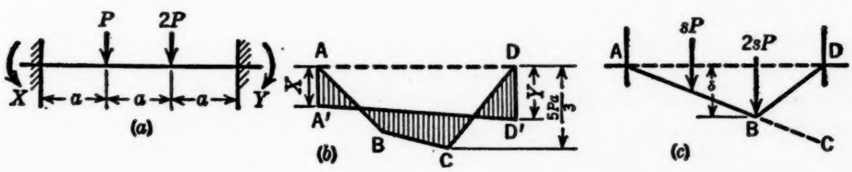

Fig. 4

which the couples X and Y could produce in this simply supported beam. Fig. 4(b) shows this superposition for arbitrarily assumed values of X and Y. The polygon ABCD corresponds to the given loads and, therefore, is fixed. The line A′D′ corresponds to the assumed values of X and Y and may be moved.

For any assumed bending moment distribution which is compatible with the given loads (that is, for any assumed position of the line A′D′, Fig. 4(*b*)), it is easy to determine the largest multiplier m that can be applied to these bending moments if the distribution is still to remain admissible. Indeed, this factor m is found by dividing the limit moment by the largest absolute value of the bending moment. Obviously, m is a statically admissible multiplier in the sense defined in connection with theorem I, and hence a lower bound for the safety factor s. Moreover, the safety factor is the largest value that can be obtained for m by changing the position of the line A′D′. To increase m, one must decrease the largest bending moment. Inspection of Fig. 4(*b*) reveals that, for the given loads, the value of the bending moment at a certain cross section can be decreased only at the cost of increasing the value of the bending moment at some other cross section. The optimum result is achieved by drawing the line A′D′ so that the bending moments at the two ends and under the load $2\,P$ become equal. A simple computation shows that this smallest possible value of the largest bending moment equals $5\,P\,a/6$. The safety factor of the beam, therefore, equals

$$s = \frac{6\,M_o}{5\,P\,a} \qquad (1)$$

The fact that the bending moments at the ends and under the load $2\,P$ had to be made equal in absolute value indicates that the yield hinges in the collapse state are located at these cross sections. Of course, in simple problems such as the one just considered, there can scarcely be any doubt concerning the location of the yield hinges. In more complicated, highly indeterminate structures, however, even an experienced stress analyst will frequently fail to guess the location of all yield hinges correctly. In such cases the principles of this section and of section 3 become really useful.

3. An Upper Bound for the Safety Factor

The principle to be established in this section furnishes the upper bound for the safety factor. By definition, such an upper bound is larger than the safety factor, or at least equal to it. By the joint use of both principles, the safety factor can therefore be bounded from above and below. This fact is of great practical importance, because the precise determination of the safety factor is unnecessary when the gap between the lower and upper bounds is sufficiently small for practical purposes.

Consider a statically indeterminate beam or frame in a state of collapse. The given loads have been multiplied by the largest statically admissible multiplier and cannot be increased any further. At the same time, the structure has been transformed into a mechanism by the appearance of a sufficient number of yield hinges, and this mechanism begins to move under the collapse loads (the given loads multiplied by the largest statically admissible multiplier). This beginning collapse can be treated as a quasi-static process:

> Theorem II.—The work that the collapse loads do on the displacements of their points of application must equal the work that the limit moments in the yield hinges do on the relative rotations of the parts connected by the hinges.

If the location of the yield hinges were known, this application of the principle of virtual work would furnish the ratio of the collapse loads to the given loads—that is, the safety factor.

To illustrate the use of this principle, consider again the beam in Fig. 4(*a*), assuming it to be known that the yield hinges in the state of collapse are located at the ends and under the larger load. Fig. 4(*c*) shows the beam just after collapse has begun. If δ denotes the infinitesimal displacement of the load $2\,s\,P$, the displacement of the load $s\,P$ is $\delta/2$ and the relative rotations at the hinges (angles DAB, CBD, and BDA in Fig. 4(*c*)) are $\frac{\delta}{2\,a}$, $\frac{\delta}{2\,a} + \frac{\delta}{a}$, and $\frac{\delta}{a}$, respectively. The work of the limit moments in the yield hinges is therefore

$$W_o = 2\,M_o\left(\frac{\delta}{2\,a} + \frac{\delta}{a}\right) \qquad (2)$$

and the work of the collapse loads is

$$W_c = 2\,s\,P\,\delta + s\,P\,\frac{\delta}{2} \qquad (3)$$

Equating these expressions and solving for s yields Eq. 1 as before.

Instead of the actual yield hinges, whose location is not known a priori, consider now any arrangement of yield hinges which transforms the structure into a mechanism. The principle of virtual work, applied in the manner described, then furnishes the multiplier n which must be applied to the given loads if the assumed yield hinges are to become active, and if no attention is paid to the fact that the limit moment might then be exceeded at some cross sections other than the assumed hinges. Any multiplier n obtained in this manner will be called "kinematically sufficient" (to produce collapse).

With this terminology, the principle for the safety factor can be stated as follows:

> Theorem III.—The safety factor against collapse is the smallest kinematically sufficient multiplier.

A proof of this principle is given in the Appendix.

4. Example

The most practical method of combining the principles of sections 2 and 3 is as follows:

(*a*) Trying to guess the actual mode of collapse, assume an arrangement of yield hinges that transforms the structure into a mechanism and determine the corresponding kinematically sufficient multiplier n. This is an upper bound for the safety factor s—that is, $s \leqq n$.

(*b*) For the collapse mode assumed in step (*a*) determine the bending moments M. These are caused by the limit moments in the yield hinges and the given loads multiplied by the factor n found in step (*a*). In practical structures, these bending moments are statically determinate as a rule; the exceptional case is discussed in section 5. Denote by r the largest value of the ratio $|M/M_o|$ occurring anywhere in the structure. Obviously, $r \geqq 1$ because $r = 1$ at the

yield hinges as assumed in step (*a*). The bending moments M/r will be admissible in the sense defined in section 2, and n/r is a statically admissible multiplier. Accordingly,

$$\frac{n}{r} \leqq s \leqq n \qquad (4)$$

In particular, if $r = 1$, the assumed mode is the actual mode of collapse and $s = n$. On the other hand, if $r > 1$ at some cross section, this usually is an indication that a yield hinge should have been assumed at this particular cross section. If the procedure is repeated with this yield hinge replacing one of the yield hinges originally assumed, closer boundaries for the safety factor are obtained, as a rule. It can be proved that this substitution can always be made in at least one manner so as to narrow the gap between the boundaries for the safety factor. However, the mathematical tools needed for this proof would exceed the scope of the present paper. Moreover, the way in which this substitution must be made in order to have the desired effect is not known a priori but must be found by trial and error. The fact that the substitution will have this effect if made properly, thus, is only of academic interest.

As an example illustrating the foregoing procedure, consider the two-story frame shown in Fig. 2. For brevity, assume that the limit moment has the same value M_o at all cross sections. An arrangement of yield hinges which transforms this frame into a mechanism and the type of displacement made possible by these hinges are shown in Fig. 3. If the infinitesimal relative rotation at yield hinge A is denoted by θ, the relative rotations at yield hinges B, C, E, F, and G have the absolute values θ, whereas the rotation at yield hinge D has the absolute value $2\,\theta$. The signs of these relative rotations are unimportant for the application of the principle of virtual work, because the signs of the limit moments in the yield hinges are always such that the work of each limit moment on the relative rotation at the corresponding yield hinge is positive. Since the sum of the absolute values of the relative rotations at all yield hinges is $8\,\theta$, the work done by the limit moments M_o at these yield hinges is $8\,M_o\,\theta$.

Inspection of Fig. 3 shows that both horizontal beams undergo purely horizontal translations, the displacement of the upper beam being $2\,h\,\theta$ and that of the lower beam $h\,\theta$. Accordingly, the horizontal load $n\,P$ does the work $2\,h\,\theta\,n\,P$, whereas the vertical load $n\,P$ does not do any work. The principle of virtual work therefore yields

$$8\,M_o\,\theta = 2\,h\,\theta\,n\,P \qquad (5)$$

and, hence,

$$n = 4\,\frac{M_o}{P\,h} \qquad (6)$$

This value of n constitutes an upper bound for the safety factor.

Fig. 5 shows the parts of the frame under the action of the multiplied loads and the limit moments at the hinges. From Eq. 6, the multiplied load $n\,P = \frac{4\,M_o}{h}$. To simplify the lettering, M_o is taken as the unit for the couples and M_o/h as the unit for the forces. This designation implies that the length h must be treated as the unit of length. The limit moments and the exterior

loads are indicated by full lines; and the forces which the parts of the frame exert on each other, by dotted lines. The sense of the limit moments follows from the sense of the relative rotations shown in Fig. 3. The forces which the parts of the frame exert on each other are obtained by applying the conditions of equilibrium to each of these parts.

Consider first the length DE. For moment equilibrium, the horizontal reaction at the lower end must have the intensity 4. Since this reaction and the exterior load 4 at hinge D satisfy the condition for horizontal equilibrium, no horizontal force is transmitted across hinge D. Horizontal equilibrium of the segment CD requires then that no horizontal force be transmitted across hinge C either, and moment equilibrium of this segment requires that the vertical

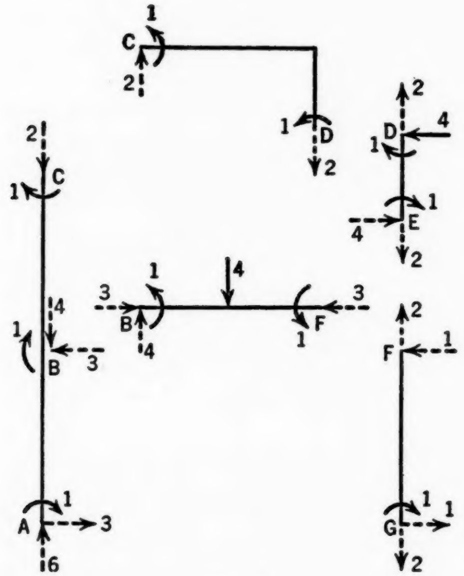

FIG. 5

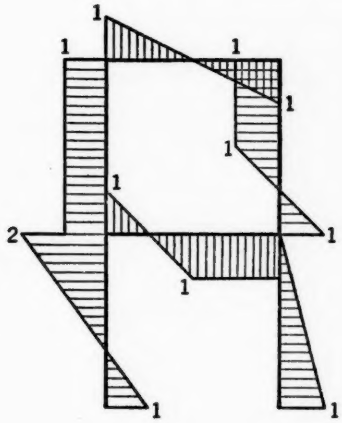

FIG. 6

forces transmitted across hinges C and D have the intensity 2. Continuing in this manner, one finds the forces indicated by dotted lines in Fig. 5. The bending moments for the members of the frame are shown in Fig. 6; the bending moments for a member are plotted perpendicularly to the axis of the member and toward that side of the member in which they produce tension. The numbers indicate the absolute values of the bending moments (in multiples of M_o).

Inspection of Fig. 6 shows that the absolute largest bending moment, occurring just below hinge B, has the intensity 2 M_o. The ratio r, therefore, has the value 2. With the value of n obtained, Eq. 4 gives

$$2\frac{M_o}{P\,h} \leqq s \leqq 4\frac{M_o}{P\,h} \quad \text{.......................(7)}$$

The bounds for the safety factor are still too far apart for practical purposes. The procedure must be repeated, therefore, with a hinge at the location of the greatest bending moment (that is, in the left-hand column, just below the first floor) replacing one of the previously assumed hinges. The decision as to which hinge is to be so replaced is somewhat facilitated by the condition that the structure must still be transformed into a mechanism by the new arrangement of hinges. Thus, it would not be admissible to replace hinge G in Fig. 3 by the new hinge.

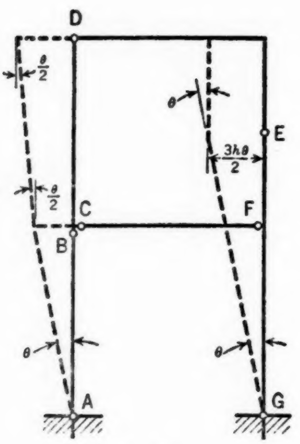

Fig. 7

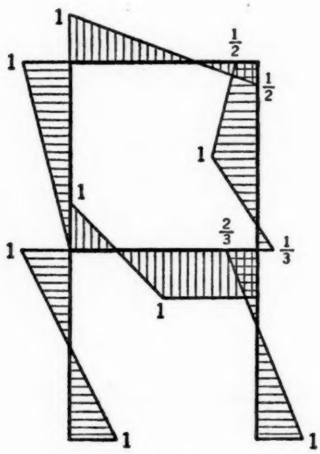

Fig. 8

Fig. 7 shows the selected arrangement of hinges and the type of displacement made possible by these hinges. With θ as defined in Fig. 7, the relative rotation at hinges A, E, F, and G has the absolute value θ; and the relative rotation at hinges B, C, and D has the absolute value $\theta/2$. The horizontal displacement of hinge E now equals $3\,\theta\,h/2$. Thus, the principle of virtual work furnishes the kinematically sufficient multiplier:

$$n = \frac{11\,M_o}{3\,P\,h} \qquad (8)$$

The value of n defined by Eq. 8 constitutes an upper bound for the safety factor. To obtain a lower bound, one must determine the bending moments corresponding to the assumed mode of collapse. Fig. 8 shows these bending moments. Since the absolute value of the bending moment nowhere exceeds the limit moment, $r = 1$, and the upper boundary is also a lower bound. Hence, the safety factor against collapse equals

$$s = \frac{11\,M_o}{3\,P\,h}, \qquad (9)$$

and Fig. 7 shows the actual mode of collapse.

5. Local Versus Total Collapse

In the preceding example the actual yield hinges are so arranged that the entire structure is transformed into a mechanism. In other cases, however, the actual yield hinges may transform only part of the structure into a mechanism without permitting any displacements of the remainder of the structure. In this section these two types of collapse will be called "total collapse" and "local collapse."

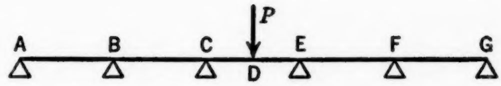

Fig. 9

For continuous beams, collapse is very often local. In fact, for a continuous beam with a single transverse load, collapse must be local. Fig. 9 illustrates this fact. For each of the segments AB, BC, $\cdots$, FG of the beam the bending moment varies linearly with the distance measured along the beam. Accordingly, the largest absolute value of the bending moment for any one of these segments can occur only at the ends of this segment. Yield hinges can thus develop only at points B, $\cdots$, F. However, yield hinges at points B and F would not contribute to the mobility of the system because of the supports there and the absence of hinges inside the adjacent spans. The only way to transform the beam into a mechanism, therefore, is to insert hinges at points C, D, and E. This arrangement of hinges, however, means local collapse.

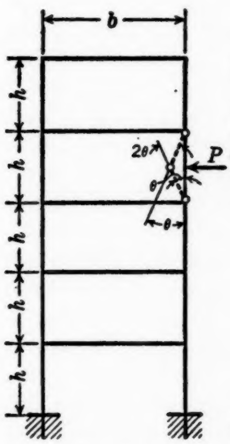

Fig. 10

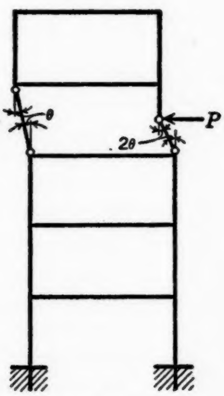

Fig. 11

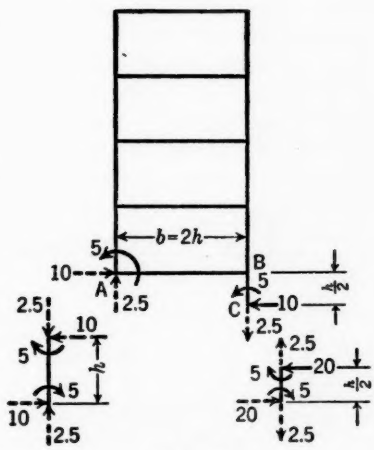

Fig. 12

For multistory frames, on the other hand, this particular type of local collapse constitutes the exception. To demonstrate this fact, consider the frame of Fig. 10. Assume that the ceiling beam and the columns of the topmost story have the limit moment M_o, the ceiling beam and the columns of the next story, the limit moment $2\ M_o$, and so forth; and, finally, the ceiling beam and the

columns of the lowest story, the limit moment $5\,M_o$. Consider, now, a single horizontal load P applied at the center of one of the columns. The dotted lines in Fig. 10 indicate the simple type of local collapse which is typical for continuous beams. The corresponding kinematically sufficient multiplier is found to be

$$n_1 = \frac{8\,k\,M_o}{P\,h} \qquad \text{(10}a\text{)}$$

if the horizontal load is applied to the kth story from the top. Fig. 11 shows a more complicated type of local collapse. The corresponding kinematically sufficient multiplier equals

$$n_2 = \frac{6\,k\,M_o}{P\,h} \qquad \text{(10}b\text{)}$$

in which k has the same meaning as Eq. 10a. Since both n_1 and n_2 are the upper bounds for the safety factor and since $n_2 < n_1$, the simple type of local collapse shown in Fig. 10 cannot occur in the frame considered here.

To show that the type of local collapse indicated in Fig. 11 can actually occur, consider the case in which the horizontal load is applied to the lowest story. The kinematically sufficient multiplier n_2 (and therefore the upper bound for the safety factor) then has the value $30\,\frac{M_o}{P\,h}$. Fig. 12 shows the couples and forces acting on the parts of the frame for the assumed mode of collapse. (As in Fig. 5, M_o is used as unit for the couples and M_o/h as unit for the forces, and b is assumed to equal $2\,h$.) To find a lower bound for the safety factor, one must determine bending moments in the structure corresponding to these couples and forces. At first glance, this might appear to be a difficult task because of the indeterminacy of the upper part of the frame. However, all that is needed is a distribution of bending moments which is "statically compatible" with these forces and couples (as explained in section 2). Such a distribution is obtained, for instance, by letting the forces and couples, which are applied to the upper part of the frame, be carried by beam AB and column BC (Fig. 12) alone. As a result, the beams and columns of the higher stories are free from bending moments. The bending moments in the lowest story, which are obtained in this manner, are shown in Fig. 13. Nowhere do they exceed the limit moment $5\,M_o$ of the members of this story. Thus, the assumed mode of collapse is the correct one and the safety factor equals $30\,M_o/P\,h$.

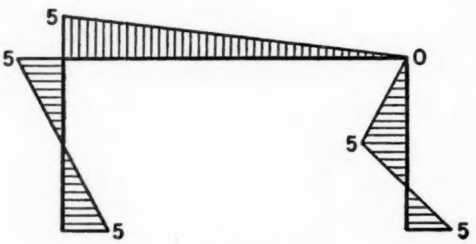

FIG. 13

ACKNOWLEDGMENT

The results presented in this paper were obtained during the course of research conducted under contract N7-onr-35806 with the Office of Naval Research, United States Department of the Navy.

APPENDIX

The characterization of the safety factor against collapse as the largest statically admissible multiplier is an immediate consequence of:

> Theorem IV.—Collapse cannot occur under the loads obtained by multiplying the given loads by a factor which is smaller than a statically admissible multiplier.

The following proof is an adaptation, to beams and frames, of a more general discussion presented by Mr. Greenberg before the Second Symposium on Plasticity at Brown University, Providence, R. I., in April, 1949.

Let m denote a statically admissible multiplier and m' a positive number which is smaller than m. According to the definition of a statically admissible multiplier, there exist bending moments which are statically compatible with the loads multiplied by m and which, at no cross section, exceed the limit moment of this cross section in absolute value. Consequently, there exists a distribution of bending moments M' which is statically compatible with the loads multiplied by m' and satisfies the continued inequality:

$$-M_o(x) < M'(x) < M_o(x) \qquad (11)$$

at any cross section x.

To prove theorem IV, suppose that, in contradiction to this theorem, collapse were to occur under the loads multiplied by m'. Then there would exist admissible bending moments $M(x)$ statically compatible with the loads multiplied by m' and reaching the limit moment at a sufficient number of cross sections to transform the structure into a mechanism. Let $x_1, x_2, \cdots, x_n$ be the cross sections at which yield hinges are active during this collapse. Then,

$$|M(x_1)| = M_o(x_1), \cdots, |M(x_n)| = M_o(x_n) \qquad (12)$$

As long as the displacements are small enough to be disregarded in establishing the equilibrium conditions for the various parts of the collapsing structure, the collapse takes place under constant forces and bending moments. Thus, no elastic deformations occur during this initial phase of the collapse and the parts into which the structure is separated by the yield hinges move as rigid bodies. At a generic instant during this initial phase, let $\theta(x_1), \cdots, \theta(x_n)$ be the relative rotations at the yield hinges. Then

$$\sum_{k=1}^{n} M(x_k)\,\theta(x_k) = W_m \qquad (13a)$$

in which the sum W_m equals the work done by the multiplied loads (multiplier m') on the displacements of their points of application since the beginning of the collapse. According to the principle of virtual work, this work can also be written as

$$W_m = \sum_{k=1}^{n} M'(x_k)\,\theta(x_k) \qquad (13b)$$

because the bending moments $M'(x)$ are statically compatible with the same loads as the bending moments $M(x)$. From the equality of Eqs. 13, it follows

that

$$\sum_{k=1}^{n} [M(x_k) - M'(x_k)]\theta(x_k) = 0 \qquad (14)$$

Now, the bending moment $M(x_k)$ must equal the limit moment $M_o(x_k)$ in absolute value. Moreover, each term of Eq. 13*a* is positive because it represents the work of the limit moment in a yield hinge on the relative rotation at this hinge.

It is readily shown that every term of Eq. 14 must be positive, too. Indeed, if $M(x_k) = M_o(x_k)$, then $\theta(x_k) > 0$ because the contribution of the hinge x_k to the sum in Eq. 13*a* must be positive. Moreover, Eq. 11 shows that $M'(x_k) < M(x_k)$ in this case. Thus, the contribution of the hinge x_k to the sum in Eq. 14 is positive. If, on the other hand, $M(x_k) = -M_o(x_k)$, then $\theta(x_k) < 0$ and $M'(x_k) > M(x_k)$. Therefore, the contribution of hinge x_k to the sum in Eq. 14 is again positive. All terms of this sum being positive, Eq. 14 cannot be fulfilled and the assumption that collapse occurs under the loads multiplied by m' has led to a contradiction. This completes the proof of the maximum principle of section 2.

In the proof of the minimum principle of section 3 the following theorem[4] is used:

> Theorem V.—If a beam or frame is strengthened (that is, if its cross sections are changed in such a manner that the limit moment is increased for, at least, one cross section and decreased for none), the safety factor for a given system of loads cannot decrease as a result of this strengthening.

Indeed, any distribution of bending moments that is admissible for the original structure is also admissible for the strengthened structure. Thus, any multiplier that is statically admissible for the original structure is also statically admissible for the strengthened structure. Consequently, the safety factor of the original structure which is the largest statically admissible multiplier for this structure is also a statically admissible multiplier for the strengthened structure. Therefore, the safety factor of the strengthened structure that is the largest statically admissible multiplier for this structure cannot be smaller than the safety factor of the original structure.

Consider, now, the manner in which a kinematically sufficient multiplier n is obtained: Yield hinges are assumed in sufficient number to transform the structure into a mechanism, and the multiplier n is determined which must be applied to the given loads if the structure is to collapse by yielding at the assumed hinges. Of course, the assumed mode of collapse may not be the actual mode of collapse for the given structure. It is easy, however, to define a structure for which the assumed mode of collapse is the actual one. Indeed, consider the structure derived from the given structure by leaving the limit moments unchanged at the assumed yield hinges and increasing them indefinitely everywhere else. The actual collapse mode of this strengthened structure will obviously be the mode used in the determination of the kinematically sufficient multiplier n. In other words, n is the safety factor of the strengthened structure and, therefore, cannot be smaller than the safety factor of the given structure. This completes the proof of the minimum principle of section 3.

[4] "The Principle of Limiting Stress," by S. M. Feinberg, *Prikladnaya Matematika i Mekhanika*, Vol. 12, 1948, pp. 63–68 (in Russian).

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